Lec 10. Uniformly Most Powerful Tests

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Learning objectives

After this section, you should be able to ...

- understand the motivation of most powerful test and randomized test
- understand the Neymann-Pearson Lemma
- find the UMP tests for specific problems

10.1 Uniformly Most Powerful Test

Let Φ_α denote all level α tests of testing problem $H_0 : \theta \in \Theta_0 \leftrightarrow H_1 : \theta \in \Theta_1$. A test $\phi \in \Phi_\alpha$ is **uniformly most powerful** (UMP) if

$$\beta_\phi(\theta) \geq \beta_{\phi_1}(\theta), \forall\ \phi_1 \in \Phi_\alpha, \forall\ \theta \in \Theta_1$$

Def.

So the general problem is: for $\alpha \in [0, 1]$, we select ϕ so as to maximize its power subject to have level α

$$\arg\max_\phi \beta_\phi(\theta) = E_\theta \phi(\mathbf{X}), \forall \theta \in \Theta_1$$

$$s.t. \quad \max_{\theta \in \Theta_0} \beta_\phi(\theta) \leq \alpha$$

Theorem 1 (Neyman-Pearson Lemma)**.** *Consider testing* $H_0 : \theta = \theta_0$ *versus* $H_1 : \theta = \theta_1$*. (Simple null and simple alternative), where the sample pdf or pmf corresponding to* θ_i *is* $f(x, \theta_i), i = 0, 1$*.*

1. Existence: for each $\alpha \in (0, 1)$ *there exists a test* ϕ *and a constant* $0 \leq c$ *and* $0 \leq r \leq 1$ *such that*

$$\phi(\mathbf{x}) = \begin{cases} 1, & f(\mathbf{x}; \theta_1) / f(\mathbf{x}; \theta_0) > c \\ r, & f(\mathbf{x}; \theta_1) / f(\mathbf{x}; \theta_0) = c \\ 0, & f(\mathbf{x}; \theta_1) / f(\mathbf{x}; \theta_0) < c \end{cases} \tag{10.1}$$

and

$$E_{\theta_0} \phi(\mathbf{X}) = \alpha. \tag{10.2}$$

Theorem 1 (Neyman-Pearson Lemma)**.**

2. Sufficient condition for a UMP test. If a test satisfies (10.1) *and* (10.2) *for some* c *and* r*, then it is UMP test for* H_0 *against* H_1 *at level* α*.*

Theorem 1 (Neyman-Pearson Lemma)**.**

3. Necessary condition for a UMP test. If $\tilde{\phi}$ is a UMP test at level α for testing H_0 against H_1, ϕ is defined by (10.1) and (10.2), then $\tilde{\phi} = \phi, a.e.$. (except on a set A such that $P_{\theta_0}(A) = P_{\theta_1}(A) = 0$.

Proof. For any test $\tilde{\phi}$ we have

$$(\phi(\mathbf{x}) - \tilde{\phi}(\mathbf{x}))(f(\mathbf{x}, \theta_1) - cf(\mathbf{x}, \theta_0)) \geq 0$$

thus

$$0 \leq \int (\phi(\mathbf{x}) - \tilde{\phi}(\mathbf{x}))(f(\mathbf{x}, \theta_1) - cf(\mathbf{x}, \theta_0))d\mathbf{x} = \beta_\phi(\theta_1) - \beta_{\tilde{\phi}}(\theta_1) - c(\beta_\phi(\theta_0) - \beta_{\tilde{\phi}}(\theta_0)).$$

Notice that ϕ is a UMP test for $H_0 \leftrightarrow H_1$, then if $\tilde{\phi}$ is also a UMP test, then $\beta_\phi(\theta_1) = \beta_{\tilde{\phi}}(\theta_1)$, we conclude that $c(\alpha - \beta_{\tilde{\phi}}(\theta_0)) \geq 0$, that is $\beta_{\tilde{\phi}}(\theta_0) \geq \alpha$. But $\beta_{\tilde{\phi}}(\theta_0) \leq \alpha$, we have $\beta_{\tilde{\phi}}(\theta_0) = \alpha$. Then

$$\tilde{\phi} = \phi, a.e.$$

□

Another proof of NP Lemma

In the method of Lagrange multipliers, the problem at hand is of the form $\max f(x)$ such that $g(x) \leq c$.

Theorem 1. *Let λ be a fixed non-negative number and let $x_o(\lambda)$ be a maximizer of*

$$M(x, \lambda) = f(x) - \lambda g(x)$$

Then $x_o(\lambda)$ maximizes $f(\mathrm{x})$ over all x such that $g(x) \leq g\left(x_o(\lambda)\right)$.

Proof. By assumption, $\lambda \geq 0$ and $x_o = x_o(\lambda)$ satisfies

$$f\left(x_o\right) - \lambda g\left(x_o\right) \geq f(x) - \lambda g(x)$$

Then $f\left(x_o\right) \geq f(x) - \lambda\left(g(x) - g\left(x_o\right)\right)$ Now let $S = \{x : g(x) \leq g\left(x_o\right)\}$. Since $\lambda \geq 0$, we conclude

$$f\left(x_o\right) \geq f(x) \qquad \forall x \in S.$$

□

Proof of NP Lemma: The problem at hand is $\max \beta_\phi(\theta_1)$ such that $\beta_\phi(\theta_0) = \alpha$. Notice that $\phi(\mathbf{x})$ is decided by the reject region R. The Lagrangian is

$$\begin{aligned} M(\phi, \lambda) &= \beta_\phi(\theta_1) - \lambda \beta_\phi(\theta_0) \\ &= \int_R f(\mathbf{x}, \theta_1)\, d\mathbf{x} - \lambda \int_R f(\mathbf{x}, \theta_0)\, d\mathbf{x} \\ &= \int_R \left[f(\mathbf{x}, \theta_1) - \lambda f(\mathbf{x}, \theta_0) \right] d\mathbf{x} \end{aligned}$$

thus the maxima is achieved with $R_o = \{\mathbf{x} : f(\mathbf{x}, \theta_1) > \lambda f(\mathbf{x}, \theta_0)\}$. That is

$$\phi_o(\mathbf{x}) = \begin{cases} 1, & f(\mathbf{x}, \theta_1) / f(\mathbf{x}, \theta_0) > \lambda \\ 0, & f(\mathbf{x}, \theta_1) / f(\mathbf{x}, \theta_0) < \lambda \end{cases}$$

Now determine λ as value that satisfies $\beta_{\phi_o}(\theta_0) = \alpha$. A randomized test will be needed for discrete cases.

10.2 Find a UMP test if existed

General procedure to find a UMP test of $H_0 : \theta \in \Theta_0 \leftrightarrow H_1 : \theta \in \Theta_1$ based on Neymann-Pearson Lemma:

1. Take $\theta_0 \in \partial\Theta_0$, and arbitrary $\theta_1 \in \Theta_1$. Then consider simple hypotheses
$$H_0' : \theta = \theta_0 \leftrightarrow H_1' : \theta = \theta_1$$

2. Find the UMP level α test ϕ_{θ_1} for $H_0' \leftrightarrow H_1'$ by Neymann-Pearson Lemma

3. Verify ϕ_{θ_1} is irrelevant to θ_1, thus, $\phi_{\theta_1} = \phi$

4. Show ϕ has level α over Θ_0

10.3 Monotone Likelihood Ratio family

The family of models (sample pdfs or pmfs) $\{f(\mathbf{x},\theta) : \theta \in \Theta\}$ with $\Theta \subset R$ is said to be a **monotone likelihood ratio (MLR) family** in T if for $\theta_1 < \theta_2$,
(1) the distribution $f(\mathbf{x},\theta_1)$ and $f(\mathbf{x},\theta_2)$ are **distinct**, and
(2) there exists a **one-dimensional** statistic $T(\mathbf{x})$ such that the ratio $f(\mathbf{x};\theta_2)/f(\mathbf{x};\theta_1)$ is a **monotone function** of $T(\mathbf{x})$.

Def.

Some examples.

Theorem 3 (Karlin-Rubin, UMP test for MLR family)**.** *Consider testing problem* $H_0 : \theta \leq \theta_0 \leftrightarrow H_1 : \theta > \theta_0$*. Suppose* $T(X)$ *is a sufficient statistic for* θ *and the family of pdfs or pmfs* $\{f(t;\theta) : \theta \in \Theta\}$ *for the sufficient statistic* T *has a nondecreasing MLR. Then for any* t_0*, the test*

$$\phi(T) = \begin{cases} 1, & T > t_0 \\ r, & T = t_0 \\ 0, & T < t_0 \end{cases}$$

is a UMP level α *test of* $H_0 \leftrightarrow H_1$*, where* $\alpha = E_{\theta_0}\phi(T)$*.*

Proof. Let $\beta_\phi(\theta) = P_\theta(T > t_0)$ be the power function of the test. Fix $\theta' > \theta_0$. Consider testing

$$H_0' : \theta = \theta_0 \leftrightarrow H_1' : \theta = \theta'$$

By NP Lemma, the UMP test of $H_0' \leftrightarrow H_1'$ is

$$\phi(\mathbf{x}) = \begin{cases} 1, & f(\mathbf{x},\theta')/f(\mathbf{x},\theta_0) > c_0 \\ r, & f(\mathbf{x},\theta')/f(\mathbf{x},\theta_0) = c_0 \\ 0, & f(\mathbf{x},\theta')/f(\mathbf{x},\theta_0) < c_0 \end{cases} \Leftrightarrow \phi(T(\mathbf{x})) = \begin{cases} 1, & T(\mathbf{x}) > t_0 \\ r, & T(\mathbf{x}) = t_0 \\ 0, & T(\mathbf{x}) < t_0 \end{cases}$$

The equivalence holds because T is sufficient and $f(t,\theta)$ has MLR. Notice r and t_0 satisfy

$$E_{\theta_0}\phi(T) = P_{\theta_0}(T > t_0) + rP_{\theta_0}(T = t_0) = \alpha,$$

we conclude ϕ is also UMP test of $H_0' \leftrightarrow H_1$.

Then we show ϕ is level α test for H_0. Since the family of pdfs or pmfs of T has an MLR, $\beta_\phi(\theta)$ is nondecreasing, so

$$\sup_{\theta \le \theta_0} \beta_\phi(\theta) = \beta_\phi(\theta_0),$$

and ϕ is a level α test for H_0. □

Corollary 1. *Under the conditions of Theorem 3, a UMP test exists for*

$$H_0 : \theta \geq \theta_0 \leftrightarrow H_1 : \theta < \theta_0$$

and has form

$$\phi(T(\mathbf{x})) = \begin{cases} 1, & T(\mathbf{x}) < t_0 \\ r, & T(\mathbf{x}) = t_0 \\ 0, & T(\mathbf{x}) > t_0 \end{cases}$$

where r *and* t_0 *satisfy* $E_{\theta_0}\phi(T) = \alpha$.